Simple example of partial synchronization of chaotic systems

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A system of three nonsymmetrically coupled skew tent maps is considered. It is shown that in a large region of the parameter space, partial chaotic synchronization takes place. This means that two variables synchronize, while the third does not synchronize with the first two, and while the global motion is chaotic. The different bifurcations that lead to this behavior, as well as to its disappearance, are discussed. [S1063-651X(98)01011-3]

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I. INTRODUCTION

The fact that two chaotic systems may synchronize while without losing their chaotic behavior is now well known, and the mechanisms of synchronization and its loss have attracted much attention in the mathematical and physical literature (e.g., Refs. [1–3]). A simple introduction based on the same type of chaotic system that is considered in this paper is given in Ref. [4]. Synchronization effects in large ensembles of coupled chaotic systems, hereafter called *cells*, have also been considered [5,6]. In large systems, usually not all cells synchronize, i.e., we have the phenomenon of partial synchronization. The purpose of this paper is to study in some detail partial synchronization in a much simpler system of just three coupled cells, with nonsymmetric coupling, where, apart from Ref. [7], partial chaotic synchronization has not been reparted so far. We think that this phenomenon will have many applications in engineering, in particular in signal precessing.

II. EXAMPLE

We consider the three-dimensional map $F:\mathbb{R}^3\to\mathbb{R}^3$

$$F:\begin{pmatrix} x\\ y\\ z\end{pmatrix}\mapsto\begin{pmatrix} f[x+\varepsilon(z-x)]\\ f[y+\varepsilon(z-y)]\\ f[z+\varepsilon(x-z)]\end{pmatrix}, \tag{1}$$

where the f is the skew-tent map defined by

$$f(x)=\begin{cases} f_1(x) = b\dfrac{x}{a} & \text{for} \quad x\leqslant a\\[2ex] f_2(x) = b\dfrac{1-x}{1-a} & \text{for} \quad x>a,\end{cases} \tag{2}$$

and with parameter values a and b restricted to

$$0<a<1, \quad \max\{a,1-a\}<b\leqslant 1. \tag{3}$$

The function f has the attracting invariant interval

$$I=[f(b),b]\subseteq[0,1], \tag{4}$$

and the one-dimensional dynamical system generated by the iterations of f on I has a chaotic behavior, because the slope of f is always greater than 1 in absolute value. Furthermore, this dynamical system has a natural invariant measure μ on I with a density $\rho(x)$. [8].

Note that the function F is not symmetrical under permutations of x,y,z. The diagonal planes in $\mathbb{R}^3$,

$$\Pi_{xy}=\{(x,y,z)|x=y\},$$
$$\Pi_{xz}=\{(x,y,z)|x=z\}, \tag{5}$$

are invariant under F, but not the plane Π_{yz}. The main diagonal

$$D=\{(x,y,z)|x=y=z\}=\Pi_{xy}\cap\Pi_{xz} \tag{6}$$

is also invariant.

Now we consider the three-dimensional dynamical system generated by the iterations of F. Restricted to D, it is equivalent to the one-dimensional system generated by the iterations of f. Restricted to Π_{xy}, it is equivalent to the two-dimensional dynamical system generated by the iterations of

$$F_{xy}=F|_{\{x=y\}}: \quad \begin{pmatrix} x\\ z\end{pmatrix}\mapsto\begin{pmatrix} f[x+\varepsilon(z-x)]\\ f[z+\varepsilon(x-z)]\end{pmatrix}, \tag{7}$$

and restricted to Π_{xz} it is equivalent to the two-dimensional dynamical system generated by the iterations of

$$F_{xz}=F|_{\{x=z\}}: \quad \begin{pmatrix} x\\ y\end{pmatrix}\mapsto\begin{pmatrix} f(x)\\ f[y+\varepsilon(x-y)]\end{pmatrix}. \tag{8}$$

The question addressed in this paper is under what conditions the trajectories will converge to Π_{xy}, to Π_{xz}, and/or to D, and thus have an asymptotic behavior governed by Eqs. (7) and (8) and the iterations of f, respectively.

III. PARTIAL CHAOTIC SYNCHRONIZATION

Partial synchronization is the phenomenon when, in a dynamical system, only part of the state variables synchronize and the others do not synchronize with them. Thus the trajectories converge to an invariant linear subspace defined by the equality of the synchronizing variables. If the motion in this subspace is chaotic, we speak of *partial chaotic synchronization*. For three-dimensional systems, two variables undergo partial chaotic synchronization, if the following conditions are satisfied.

(1) The corresponding plane is invariant under the action of the map.

(2) The plane is stable under three-dimensional dynamics.

(3) The diagonal of the plane is transversally repelling, in order to avoid total synchronization.

(4) The motion in the plane is chaotic.

In our three-dimensional system the invariant linear subspaces are Π_{xy} and Π_{xz}. Thus partial chaotic synchronization may take place between x and y, or between x and z.

Let us remark that the coexistence of total and partial synchronization is also possible. In this case there are the first attractor on the diagonal and the second attractor(s) outside the diagonal in the invariant plane.

IV. ASYMPTOTIC STABILITY OF THE INVARIANT PLANES

Proposition. With respect to the three-dimensional dynamics generated by the iterations of the map defined in Eqs. (1) and (2), and in the parameter range (3), (a) the plane Π_{xy} is globally asymptotically stable if and only if

$$1-\frac{1}{C}<\varepsilon<1+\frac{1}{C}, \tag{9}$$

and (b) the plane Π_{xz} is globally asymptotically stable if and only if

$$\frac{1}{2}-\frac{1}{2C}<\varepsilon<\frac{1}{2}+\frac{1}{2C}, \tag{10}$$

where

$$C=\max\left\{\frac{b}{a},\frac{b}{1-a}\right\}. \tag{11}$$

Proof: For any trajectory $\{(x(k),y(k),z(k))|k=0,1,2,\dots\}$, we have

$$\begin{aligned}|x(k+1)-y(k+1)|&=|f(x(k)+\varepsilon[z(k)-x(k)])-f(y(k)\\&\quad+\varepsilon[z(k)-y(k)])|\\&\leqslant\max_x|f'(x)||1-\varepsilon||x(k)-y(k)|\\&=C|1-\varepsilon||x(k)-y(k)|.\end{aligned}$$

Thus $|x(k)-y(k)|$ converges to zero if $C|1-\varepsilon|<1$, which is equivalent to condition (9). Furthermore, it is easy to show that at the moment $\varepsilon=1\pm(1/C)$, the fixed point for the map f on the diagonal becomes unstable in the direction transversal to the plane, which causes the loss of Lyapunov stability of the plane. The proof of part (b) is analogous. Note that in the intersection $\varepsilon\in[1-(1/C),(1/2)+(1/2C)]$ of the two regions from (a) and (b), the diagonal D is globally asymptotically stable, i.e., we have total synchronization.

V. REPELLING OF THE DIAGONAL

As a measure for the repelling of the diagonal D we consider the Lyapunov exponents of trajectories in D. The Jacobian matrix of F in a point (x,x,x) of D is

$$J=f'(x)\begin{pmatrix}1-\varepsilon & 0 & \varepsilon\\ 0 & 1-\varepsilon & \varepsilon\\ \varepsilon & 0 & 1-\varepsilon\end{pmatrix}. \tag{12}$$

Its eigenvectors and eigenvalues are

$$\begin{aligned}\nu_D&=f'(x),\quad \boldsymbol{e}_D=(1,1,1),\\ \nu_{xy}&=f'(x)(1-2\varepsilon),\quad \boldsymbol{e}_{xy}=(1,1,-1),\\ \nu_{xz}&=f'(x)(1-\varepsilon),\quad \boldsymbol{e}_{xz}=(0,1,0).\end{aligned} \tag{13}$$

Note that $\boldsymbol{e}_D$ lies in D, $\boldsymbol{e}_{xy}$ lies in Π_{xy}, and $\boldsymbol{e}_{xz}$ lies in Π_{xz}. Accordingly, the Lyapunov exponents of a trajectory in D are as follows.

For deviations within the line D,

$$\lambda_D=\lim_{K\to\infty}\frac{1}{K}\sum_{k=0}^{K-1}\ln|f'(x(k))|, \tag{14}$$

and the same as for the one-dimensional dynamics.

For deviations within the plane Π_{xy} transversal to D,

$$\lambda_{xy}=\lambda_D+\ln|1-2\varepsilon|. \tag{15}$$

For deviations within the plane Π_{xz} transversal to D,

$$\lambda_{xz}=\lambda_D+\ln|1-\varepsilon|. \tag{16}$$

Note that the value of λ_D, and thus of all Lyapunov exponents, depends on the trajectory. However, for (Lebesgue) almost all trajectories its value is

$$\begin{aligned}\overline{\lambda}_D&=\int_{f(b)}^{b}\ln|f'(x)|\rho(x)dx\\&=\int_{f(b)}^{a}\ln\left(\frac{b}{a}\right)\rho(x)dx+\int_{a}^{b}\ln\left(\frac{b}{1-a}\right)\rho(x)dx.\end{aligned} \tag{17}$$

We will distinguish two forms of repelling of the diagonal D within Π_{xy}.

(i) *Weak repelling*. if for almost all trajectories in D we have $\lambda_{xy}>0$, which is the case if

$$\overline{\lambda}_{xy}=\overline{\lambda}_D+\ln|1-2\varepsilon|>0 \tag{18}$$

The changing of the sign of this Lyapunov exponent from negative to positive is called the *blowout bifurcation*.

Strong repelling, if for all trajectories in D we have $\lambda_{xy}>0$. In this case, the attractor on D is a transversally repelling *chaotic saddle* [2], i.e., it attracts only points from D itself and its preimages.

Similar conditions can be given for weak and strong repelling of D within Π_{xz}.

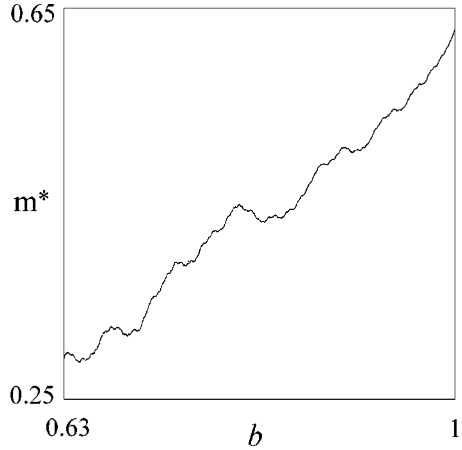

FIG. 1. The measure $m^* = m^*(b)$ of the interval $[f(b),a]$ for the skew tent map (2) at $a = 0.63$.

A. Weak repelling

The value of $\overline{\lambda_D}$ depends on the invariant density $\rho(x)$, which can only be determined by simulation, except for $b = 1$, where it is constant [4], and at some other values of b, for example, at homoclinic bifurcations of unstable periodic points. Actually, only the measure $m^* = \int_{f(b)}^{a} \rho(x) dx$ of the interval $[f(b),a]$ matters:

$$\overline{\lambda_D} = m^* \ln\left(\frac{b}{a}\right) + (1 - m^*) \ln\left(\frac{b}{1-a}\right). \tag{19}$$

This measure m^* is represented in Fig. 1 as a function of b for $a = 0.63$.

B. Strong repelling

Here we suppose that $a > 0.5$. The case $a < 0.5$ can be treated similarly. In order to find the condition for strong repelling, the trajectory with the smallest Lyapunov exponent λ_D has to be found. This can be achieved using symbolic dynamics [8]. It is the trajectory whose symbolic sequence has the largest proportion of L's with respect to R's. Its symbolic sequence is of the form

$$(L^{N-1}R)^{\infty} = \underbrace{LL \ldots L}_{N-1} R \underbrace{LL \ldots L}_{N-1} R \ldots \tag{20}$$

which corresponds to a periodic trajectory with $N-1$ points in the interval $[f(b),a]$ and one point in the interval $[a,b]$. The bifurcation when this trajectory appears takes place at the parameter values when the trajectory originated from $x = b$ passes through the critical point $x = a$. The corresponding parameter relation for the appearance of the trajectory is

$$f_1^{N-1}(f_2(b)) = b. \tag{21}$$

This leads to the following condition for the chaotic attractor on D to be a transversally repelling chaotic saddle in Π_{xy}:

$$|\varepsilon - 0.5| > \frac{[a^{N-1}(1-a)]_{1/N}}{2b}. \tag{22}$$

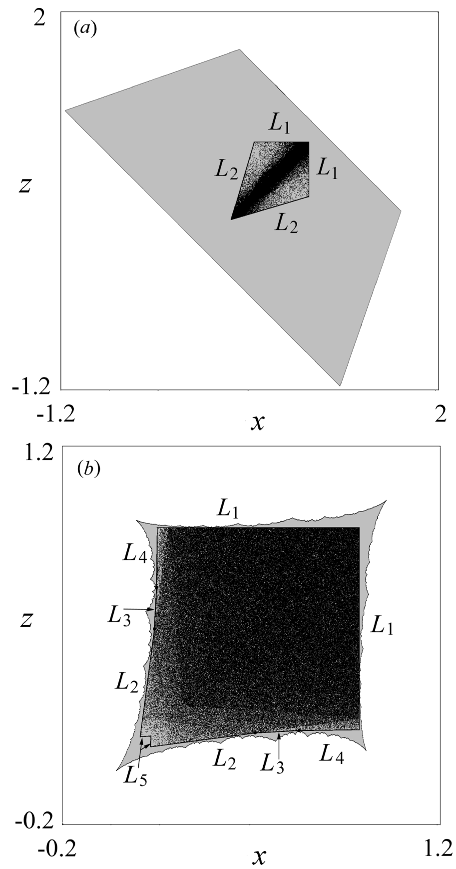

FIG. 2. Chaotic attractor (black) and its basin of attraction (gray) for different values of parameters ε at $a = 0.63$ and $b = 0.9$: (a) $\varepsilon = 0.77$—just after blowout bifurcation of the attractor on the diagonal; (b) $\varepsilon = 1.085$—just before the boundary crisis. The attractor coincides with absorbing area with boundary created by segments L_k, belonging to the iterations of critical lines given (23). k indicates the number of iteration.

VI. MOTION IN THE PLANE

We shall discuss only the motion in the plane Π_{xy}. The analysis for the plane Π_{xz} is analogous. After the blowout bifurcation, when the diagonal D becomes transversally unstable in the plane Π_{xy}, all trajectories that are repelled from D will remain bounded because of the existence of an *absorbing area* [9,10]. The boundary of this area that is invariant under the action of F_{xy} is formed by a finite number of iterations of *critical lines*, i.e., the two lines where the linear regions of the piecewise linear map F_{xy} join:

$$L_0: \quad z = -\frac{1-\varepsilon}{\varepsilon} x + \frac{a}{\varepsilon}, \quad z = -\frac{\varepsilon}{1-\varepsilon} x + \frac{a}{1-\varepsilon}. \tag{23}$$

Examples of absorbing areas which are at the same time chaotic attractors, and their basins of attraction, are given in Fig. 2 for two different values of parameters.

Increasing ε the boundary of the basin of attraction touches the attractor which leads to its destruction. The trajectories then diverge to infinity. The numerically computed curve of this boundary crisis is shown in Fig. 3. It is the right border of the domain of partial synchronization.

In addition, in Fig. 3 an area is horizontally hatched where

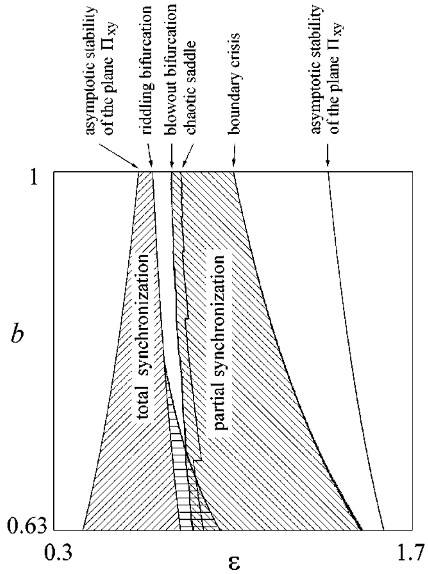

FIG. 3. Regions in parameter (ε,b) plane in which the regime of chaotic total $x=y=z$ or chaotic partial synchronization $x=y\neq z$ are asymptotically stable in three-dimensional state space; $a=0.63$.

in the plane Π_{xy} two chaotic attractors outside of the diagonal exist. Before blowout bifurcation, the basin of attractor on the diagonal becomes globally riddled [11] with the basins of these two additional attractors.

VII. CONCLUSION

We have presented detailed results on the behavior of three nonsymmetrically coupled skew-tent maps. We have shown that, in a large region of the parameter space, the phenomenon of partial chaotic synchronization takes place, i.e., a chaotic motion where two of the three state variables synchronize, whereas the third does not synchronize with the other two. The various bifurcations that take place have been identified in the ε-b parameter plane for fixed a, where ε is the coupling constant, a the breakpoint of the skew tent map and b its maximum. The existence of a region in this parameter plane is also shown where there is coexistence of total synchronization and partial synchronization, depending on the initial condition.

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[1] J.C. Alexander, J.A. Yorke, and Z. You, Int. J. Bifurcation Chaos Appl. Sci. Eng. **2**, 795 (1992).

[2] P. Ashwin, J. Buescu, and I. Stewart, Nonlinearity **9**, 703 (1996).

[3] Yu. Maistrenko and T. Kapitaniak, Phys. Rev. E **54**, 3285 (1996).

[4] M. Hasler and Yu. Maistrenko, IEEE Trans. Circuits Syst., I: Fundam. Theory Appl. **44**, 856 (1997).

[5] K. Kaneko, Physica D **41**, 137 (1990); **103**, 505 (1997).

[6] D. H. Zanette and A. S. Mikhailov, Phys. Rev. E **57**, 276 (1998).

[7] K. Pyragas, Phys. Rev. E **54**, R4508 (1996); M. S. Vieira and A. J. Lichtenberg, *ibid.* **56**, R3741 (1997).

[8] W. de Melo and S. van Strien, *One-Dimensional Dynamics* (Springer-Verlag, Berlin, 1993).

[9] Yu. L. Maistrenko, V. L. Maistrenko, A. Popovich, and E. Mosekilde, Phys. Rev. Lett. **80**, 1638 (1998).

[10] C. Mira, L. Gardini, A. Barngola, and J.-C. Cathala, *Chaotic Dynamics in Two-Dimensional Noninvertible Maps* (World Scientific, Singapore, 1996).

[11] Yu. Maistrenko, T. Kapitaniak, and P. Szuminski, Phys. Rev. E **56**, 6393 (1997).